Powerful 2-Engel groups

Primož Moravec
Oddelek za Matematiko
Inštitut za Matematiko, Fiziko in Mehaniko
Jadranska 19
1000 Ljubljana, Slovenia
email: primoz.moravec@fmf.uni-lj.si

Gunnar Traustason
Department of Mathematical Sciences, University of Bath,
Bath BA2 7AY, UK
email: gt223@maths.bath.ac.uk

We study powerful 2-Engel groups. We show that every powerful 2-Engel group generated by three elements is nilpotent of class at most two. Surprisingly the result does not hold when the number of generators is larger than three. In this paper and its sequel we classify powerful 2-Engel groups of class three that are minimal in the sense that every proper powerful section is nilpotent of class at most two.

1 Introduction

Every finite n-Engel group is nilpotent [7]. However if $n \geq 3$, the class is not n-bounded. In contrast we know that the class is n-bounded if one adds the further condition that the group is powerful [1]. (Recall that a finite p-group, p odd, is said to be powerful if $[G, G] \leq G^p$. We refer to [6] for further information and description of their many abelian like properties). The proof of this result relies on deep results on Lie algebras. It does not give any precise information how the property of being powerful affects the structure of the group and in particular we have no good bounds for the nilpotence class, not even for small values of n. In this paper and its sequel we carry out a detailed analysis of powerful 2-Engel groups.

We recall that a group is said to be 2-Engel if it satisfies the commutator law $[[y, x], x] = 1$ or equivalently the law $[x^y, x] = 1$, i.e. any two conjugates commute. These groups have their origin in Burnside's paper [2]. In that paper Burnside proved that all groups of exponent three are locally finite and in order to obtain this result he observed that these groups are 2-Engel. Burnside wrote a sequel to this paper where he studies 2-Engel groups in general [3]. This paper seems to have received surprisingly little attention, being the first paper written on Engel groups. In this paper Burnside proves that any 2-Engel group satisfies the laws

$$\begin{aligned} [x, y, z] &= [y, z, x] && (1) \\ [x, y, z]^3 &= 1. && (2) \end{aligned}$$

In particular every 2-Engel group without elements of order 3 is nilpotent of class at most 2. Burnside failed however to observe that these groups are in general nilpotent of class at most 3, although he proved (in modern terminology) that any periodic 2-Engel group is locally nilpotent. It was C. Hopkins [4] that seems to have been the first to show that the class is at most 3. So any 2-Engel group also satisfies

$$[x, y, z, t] = 1. \tag{3}$$

Hopkins also observes that (1)-(3) characterize 2-Engel groups. This transparent description of the variety of 2-Engel groups is usually attributed to Levi [5], although his paper appears much later.

Of course this settles the study of 2-Engel groups no more than knowing that the variety of abelian groups is characterized by the law $[x, y] = 1$ settles the study of abelian groups. For example, the following well known problems raised by Caranti still remain unsolved.

Problem. (a) Let G be a group of which every element commutes with all its endomorphic images. Is G nilpotent of class at most 2?

(b) Does there exist a finite 2-Engel 3-group of class three such that $\mathrm{Aut}\, G = \mathrm{Aut}_c G \cdot \mathrm{Inn}\, G$, where $\mathrm{Aut}_c G$ is the group of central automorphisms of G?

The class of powerful p-groups is quite a special class of p-groups. In some sense the groups are very abelian like and share many of the properties that abelian groups have. These groups are however also at the same time quite typical p-groups and generate for example the variety of all groups as they satisfy no non-trivial group law. For this reason one is often able to reduce problems on p-groups to the class of powerful p-groups where one can make use of all the abelian like properties. Our belief is therefore that understanding the structure of powerful 2-Engel groups should be helpful in tackling the various open problems on 2-Engel groups like the problems mentioned above.

Let us now describe the main results of this paper. We mentioned above that every 2-Engel group is nilpotent of class at most 3 and the third term of the lower central series has exponent dividing 3. In particular, it follows that in the case when a 2-Engel group has no elements of order 3, we get the best possible bound for the nilpotence class, namely 2. From Burnside we know that every group of exponent 3 is a 2-Engel group and one can think of these groups in some sense as the generic examples of 2-Engel groups where the class goes up to 3. Any powerful 3-group of exponent 3 is however abelian. One might therefore expect that all the examples of class 3 would disappear if we add the condition that the 2-Engel group is powerful. We will see that this is true when the group is generated by three elements. Surprisingly this is not true however in general. We will classify the minimal powerful 2-Engel groups, where by minimal we mean that all proper powerful sections have class at most 2. For a powerful 3-group G we have that $[G,G] \leq G^3$ and therefore $\gamma_3(G) \leq [G,G]^3$. It will be useful to divide the minimal counterexamples into two subclasses.

(I) The minimal examples G where $\gamma_3(G) < [G,G]^3$.
(II) The minimal examples G where $\gamma_3(G) = [G,G]^3$.

In this paper we deal only with the minimal examples of type I. We will give a concrete classification by listing them all as three infinite families, one with groups of rank 5 and two with groups of rank 4. It turns out that the second case is very different and those of type II will use a different approach and will be dealt with in another paper. The minimal examples of type II will form an infinite family, described in terms of irreducible polynomials over the field of three elements and including examples of any even rank greater than or equal to 4, plus one isolated minimal group of rank 5. It is a curious fact that the only minimal examples of odd rank are those of rank 5.

It had been our hope that within this rich class of minimal examples we would find some counter examples to the problems above raised by Caranti. This turns out not to be the case. It seems to suggest that such counter examples do not exists and perhaps there is a way of reducing these problems to powerful 2-Engel groups.

2 Powerful 2-Engel groups generated by three elements

We show that every 3-generator powerful 2-Engel group is nilpotent of class at most 2. Notice that the class of powerful p-groups is not closed under taking subgroups and it does not follow from this result that all powerful 2-Engel groups are nilpotent of class at most 2. In fact we will see later that there is a rich class of counter examples.

Theorem 2.1 *Every 3-generator powerful 2-Engel group is nilpotent of class at most 2.*

Proof We argue by contradiction and let G be a counter example of minimal order. By minimality, $Z(G)$ is cyclic (any quotient of a powerful group is powerful), and since G is 2-Engel we know that $[G,G]^3 \leq Z(G)$. Suppose that $[G,G]^3 = \langle z\rangle$. Without loss of generality, we can choose the generators x_1, x_2, x_3 of G so that

$$\begin{aligned}
[x_2,x_1]^3 &= z \qquad (4)\\
[x_3,x_1]^3 &= z^{m_1}\\
[x_3,x_2]^3 &= z^{m_2}
\end{aligned}$$

for some integers m_1, m_2. Since G is 2-Engel, we know that $[x_1,x_2,x_3] = [x_2,x_3,x_1] = [x_3,x_1,x_2]$ and that the third term of the lower central series is

$\langle[x_1, x_2, x_3]\rangle$. By our assumption $[x_1, x_2, x_3] \neq 1$.

Since G is powerful, we have an equation of the form

$$[x_2, x_1] = x_1^{3\alpha_1} x_2^{3\alpha_2} x_3^{3\alpha_3} \tag{5}$$

for some integers $\alpha_1, \alpha_2, \alpha_3$. Using the 2-Engel property, (4) and (5) we deduce that

$$\begin{aligned} 1 &= [x_2, x_1, x_1] \\ &= [x_2, x_1]^{3\alpha_2} [x_3, x_1]^{3\alpha_3} \\ &= z^{\alpha_2 + m_1\alpha_3}. \end{aligned}$$

$$\begin{aligned} 1 &= [x_2, x_1, x_2] \\ &= [x_2, x_1]^{-3\alpha_1} [x_3, x_2]^{3\alpha_3} \\ &= z^{-\alpha_1 + m_2\alpha_3}. \end{aligned}$$

But then (using (5))

$$\begin{aligned} [x_2, x_1, x_3] &= [x_3, x_1]^{-3\alpha_1} [x_3, x_2]^{-3\alpha_2} \\ &= z^{-m_1\alpha_1 - m_2\alpha_2} \\ &= (z^{-\alpha_1 + m_2\alpha_3})^{m_1} \cdot (z^{\alpha_2 + m_1\alpha_3})^{-m_2} \\ &= 1. \end{aligned}$$

This contradicts the assumption that $\gamma_3(G) \neq \{1\}$. $\Box$

3 Minimal counterexamples I

In this section we start the classification of the minimal examples of class three. As we said in the beginning we will here only deal with the minimal examples of type I. We will see that these are all of rank either 4 or 5 and that these form three infinite families.

3.1 Reduction to 4 or 5 generators

Let G be a minimal group of type I. By minimality we have that the center of G is cyclic and thus in particular we have that $[G,G]^3 = \langle z \rangle$ for some $z \in Z(G)$. By Theorem 2.1, G has rank r at least 4. For a right choice of generators, we have

$$G/[G,G] = \langle \bar{x_1} \rangle \times \langle \bar{x_2} \rangle \times \cdots \times \langle \bar{x_r} \rangle \tag{6}$$

with $o(\bar{x_1}) \geq o(\bar{x_2}) \geq \cdots \geq o(\bar{x_r})$, where $\bar{x_i}$ is the image of x_i in $G/[G,G]$.

Let $H_i = \langle x_i, \ldots, x_r \rangle [G,G]$. Notice that (6) still holds if x_i is replaced by $x_i h_i$ for any $h_i \in H_{i+1}$. We claim that with such changes, we can assume that

$$\begin{aligned} [x_1, x_2, x_3] &\neq 1 && (7) \\ [x_2, x_1]^3 &= z. && (8) \end{aligned}$$

First we turn to (7). As $\langle G \setminus H_2 \rangle = G$ we can, by replacing x_1 by some $x_1 h_1$ with $h_1 \in H_2$ if necessary, assume that $x_1 \notin Z^2(G)$. Similarly as $H_2 = \langle H_2 \setminus H_3 \rangle$, by replacing x_2 by some $x_2 h_2$ with $h_2 \in H_3$, we can assume that $[x_1, x_2] \notin Z(G)$. Notice that as G is 2-Engel there must be some $u_3 \in H_3$ for which $[x_1, x_2, u_3] \neq 1$. As $H_3 = \langle H_3 \setminus H_4 \rangle$, by replacing x_3 by some $x_3 h_3$ with $h_3 \in H_4$ we can assume that $[x_1, x_2, x_3] \neq 1$.

We now turn to (8). Since $[G,G]^3 = \langle z \rangle$ there must be some $u \in G$ such that $[u^3, G] \not\leq \langle z^3 \rangle$. As $\gamma_3(G)^3 = \{1\}$ it follows that $[(uv)^3, G] = [u^3, G][v^3, G]$. If $[x_1^3, G] \leq \langle z^3 \rangle$ it therefore follows that $[h^3, G] = \langle z \rangle$ for some $h \in H_2$. Now one of $[x_1 h, x_2, x_3], [x_1 h^{-1}, x_2, x_3]$ is non-trivial since otherwise we would get the contradiction that $[x_1, x_2, x_3] = 1$. Without loss of generality we can assume that $[x_1 h, x_2, x_3] \neq 1$. Then also $[(x_1 h)^3, G] \not\leq \langle z^3 \rangle$. By replacing x_1 by $x_1 h$ we have that both (7) and $[x_1^3, G] \not\leq \langle z^3 \rangle$ are satisfied. Similarly by replacing x_2 by some $x_2 k$ or $x_2 k^{-1}$ if necessary, we can assume that $[x_2, x_1]^3 \notin \langle z^3 \rangle$ while (7) still holds. By replacing x_1 by a suitable power of itself we can assume that $[x_2, x_1]^3 = z$. We have thus seen that x_1, x_2, x_3 can be chosen such that both (7) and (8) hold.

Let

$$H = \sqrt[3]{[G,G]} = \{u \in G \,:\, u^3 \in [G,G]\}.$$

As G is powerful, we have that $[G,G]=H^3$. From this fact and (6) it follows that

$$H=\langle x_1^{3^{n_1-1}}, x_2^{3^{n_2-1}}, \ldots, x_r^{3^{n_r-1}}\rangle$$

where $3^{n_i}=o(\bar{x}_i)$. If $H\leq K\leq G$ then K is powerful since $[K,K]\leq [G,G]=H^3\leq K^3$. In particular it follows that the group

$$K=\langle x_1, x_2, x_3, x_4^{3^{n_4-1}}, \ldots, x_r^{3^{n_r-1}}\rangle$$

is powerful and nilpotent of class 3. By the minimality of G, we must have $K=G$ and thus $n_4=\ldots=n_r=1$. Hence it follows also that $x_4^3,\ldots,x_r^3\in [G,G]$ and thus

$$[x_4,g]^3,\ldots,[x_r,g]^3\in\gamma_3(G)$$

for all $g\in G$. Let

$$R=\{a\in G\,:\,[a,g]^3\in\gamma_3(G)\text{ for all }g\in G\}.$$

Notice that R is a normal subgroup of G and that we have just seen that $x_4,\ldots,x_r\in R$. As $[G,G]^3>\gamma_3(G)$ we must have that $o(z)=3^m$ where $m\geq 2$ and since $\gamma_3(G)^3=\{1\}$ we also have $\gamma_3(G)=\langle z^{3^{m-1}}\rangle$. This m will remain fixed for the rest of the paper. Notice that $[G,G]^{3^m}\leq\langle z^{3^{m-1}}\rangle$ and thus $G^{3^{m-1}}\leq R$. We conclude that

$$\langle x_4,\ldots,x_r\rangle G^{3^{m-1}}\leq R.$$

For $4\leq i\leq r$, we know that $[x_i,x_1]^3,[x_i,x_2]^3\in\gamma_3(G)=\langle z^{3^{m-1}}\rangle$. Also $[x_2,x_1]^{3^m}=z^{3^{m-1}}$. This implies that by replacing x_i by a suitable $x_ix_1^{3^{m-1}\alpha_i}x_2^{3^{m-1}\beta_i}$ we can assume that

$$[x_i,x_1]^3=[x_i,x_2]^3=1,\quad 4\leq i\leq r. \tag{9}$$

As $G^{3^{m-1}}\leq R$ we still have that $x_4,\ldots,x_r$ are in R after these replacements. It is possible that (6) is not valid anymore but we still have that (7),(8) are valid.

We next replace x_3 by some $x_3x_1^{\alpha}x_2^{\beta}$ to enforce

$$[x_3,x_1]^3=[x_3,x_2]^3=1. \tag{10}$$

Notice that after this change $x_3 \in R$ since $[x_3, x_i]^3 \in \gamma_3(G)$ for $4 \leq i \leq r$. Using (7) we can now without violating any of (7)-(10) replace each x_i, $4 \leq i \leq r$ by some $x_i x_3^{\gamma_i}$ to enforce further that

$$[x_1, x_2, x_i] = 1, \quad 4 \leq i \leq r. \tag{11}$$

By replacing x_3 by a suitable power, we can assume that $[x_1, x_2, x_3] = z^{3^{m-1}}$. It is time to summarize. We have shown that we can choose the generators $x_1, \ldots, x_r$ of G such that

$$\begin{array}{rcl} [x_1, x_2, x_3] & = & z^{3^{m-1}} \\ [x_1, x_2, x_i] & = & 1, \quad 4 \leq i \leq r \\ [x_2, x_1]^3 & = & z \\ [x_i, x_1]^3 = [x_i, x_2]^3 & = & 1, \quad 3 \leq i \leq r \\ [x_j, x_i]^3 & \in & \langle z^{3^{m-1}} \rangle, \quad 3 \leq i < j \leq r. \end{array} \tag{A}$$

Let $S = \langle x_4, x_5, \ldots, x_r \rangle$. We want to clarify the structure from the last line of (A) further. We can think of $V = S/S^3[S,S]$ as a vector space over the field of three elements and we can identify $W = \langle z^{3^{m-1}} \rangle$ with the field of three elements. This leads to the alternating form

$$\Phi : V \times V \to W, \ (\bar{a}, \bar{b}) \mapsto [a, b]^3.$$

From the properties of G one sees readily that this is well defined. Choosing a standard basis with respect to this alternating form, we can replace the generators of S such that the following holds (for some $1 \leq n \leq \frac{r-1}{2}$)

$$\begin{array}{rcl} [x_5, x_4]^3 & = & z^{3^{m-1}} \\ & \vdots & \\ [x_{2n+1}, x_{2n}]^3 & = & z^{3^{m-1}} \\ [x_u, x_v]^3 & = & 1 \quad \text{otherwise.} \end{array} \tag{12}$$

Replacing x_3 by a suitable $x_3 x_4^{\alpha_4} \cdots x_{2n+1}^{\alpha_{2n+1}}$, we can assume that

$$[x_3, x_1]^3 = [x_3, x_2]^3 = \ldots = [x_3, x_{2n+1}]^3 = 1.$$

But $x_3^3 \notin Z(G)$, since otherwise $x_3 \in C_G(G^3)$ and as G is powerful this would lead to the contradiction that $[x_1, x_2, x_3] = 1$. Hence $2n+1 < r$ and we can after rearranging the generators assume that

$$[x_3, x_{2n+2}]^3 = z^{3^{m-1}}.$$

Replacing $x_i,\ 2n+3 \leq i \leq r$, by the appropriate $x_i x_{2n+2}^{\beta_i}$, we can assume that

$$[x_3, x_{2n+3}]^3 = \ldots = [x_3, x_r]^3 = 1.$$

Notice that this does not affect (A) or (12). By reordering the generators of G, we see that they can be chosen such that

$$\begin{array}{rcl} [x_1, x_2, x_3] & = & z^{3^{m-1}} \\ [x_1, x_2, x_i] & = & 1, \quad 4 \leq i \leq r \\ [x_2, x_1]^3 & = & z \\ [x_i, x_1]^3 = [x_i, x_2]^3 & = & 1, \quad 3 \leq i \leq r \\ [x_4, x_3]^3 & = & z^{3^{m-1}} \\ & \vdots & \\ [x_{2n+2}, x_{2n+1}]^3 & = & z^{3^{m-1}} \\ [x_j, x_i]^3 & = & 1 \quad \text{otherwise} \end{array} \qquad \text{(B)}$$

where $1 \leq n \leq \frac{r-2}{2}$ and $o(z) = 3^m$ where $m \geq 2$.

The next aim is to show that $n = 1$. First we need few lemmas.

Lemma 3.1 *We have that the subgroups $\langle x_1^3, x_2, x_3, \ldots, x_r\rangle$ and $\langle x_1, x_2^3, x_3, \ldots, x_r\rangle$ are powerful.*

Proof Firstly, for any $1 \leq i < j \leq r$, we have that if (using the fact that G is powerful)

$$[x_j, x_i] = x_1^{3\alpha_1} \cdots x_r^{3\alpha_r}$$

then modulo $\langle z^3\rangle$ we get using (B) that

$$1 \equiv [x_j, x_i, x_2] = [x_2, x_1]^{-3\alpha_1} = z^{-\alpha_1}.$$

This implies that $3|\alpha_1$ and therefore

$$[x_j, x_i] \in \langle x_1^3, x_2, x_3, \ldots, x_r\rangle^3.$$

This shows that the first subgroup of the lemma is powerful. Similar argument shows that the same is true for the latter. □

In particular, by the minimality of G, it follows from the lemma that these subgroups are nilpotent of class at most 2.

Corollary 3.2 *Let $1 \leq i < j < k \leq r$. Then the subgroup $\langle x_i, x_j, x_k \rangle$ is nilpotent of class at most 2 if $\{i, j, r\} \neq \{1, 2, 3\}$.*

Proof If $1 \notin \{i, j, k\}$ or $2 \notin \{i, j, k\}$ then it follows from Lemma 3.1 that $\langle x_i, x_j, x_k \rangle$ is nilpotent of class at most 2. We are left with the group $\langle x_1, x_2, x_i \rangle$ where $i \geq 4$. But it follows from (B) that this group is also nilpotent of class at most 2. This finishes the proof. □

Proposition 3.3 *We have that $n = 1$ and that $4 \leq r \leq 5$. Furthermore we can choose the generators such that the following relations hold:*

$$\begin{aligned}
[x_1, x_2, x_3] &= z^{3^{m-1}} \\
[x_1, x_2, x_i] &= 1, \quad 4 \leq i \leq r \\
[x_i, x_j, x_k] &= 1, \quad \{i, j, k\} \subseteq \{1, \ldots, r\}, \{i, j, k\} \neq \{1, 2, 3\} \\
[x_2, x_1]^3 &= z \\
[x_i, x_1]^3 = [x_i, x_2]^3 &= 1, \quad 3 \leq i \leq r \\
[x_4, x_3]^3 &= z^{3^{m-1}} \\
x_r^3 &\in Z(G), \quad r > 4.
\end{aligned}$$

Here $o(z) = 3^m$ where $m \geq 2$.

Proof First we show that $n = 1$. We argue by contradiction and suppose that $n \geq 2$. We claim that

$$L = \langle x_1, x_2, x_3, x_4, x_5^3, x_6, \ldots, x_r \rangle$$

is powerful. To see this, let $1 \leq i < j \leq r$ and use the fact that G is powerful to get an equation of the form

$$[x_j, x_i] = x_1^{3\alpha_1} \cdots x_5^{3\alpha_5} \cdots x_r^{3\alpha_r}.$$

Then using Corollary 3.2 and (B) we conclude that

$$1 = [x_j, x_i, x_6] = z^{-3^{m-1}\alpha_5}$$

which implies that α_5 must be divisible by 3 and thus $[x_j, x_i] \in L^3$. This shows that L is powerful and by minimality of G we then must have that L is nilpotent of class at most 2 and we get the contradiction that $[x_1, x_2, x_3] = 1$. Hence we must have that $n = 1$.

It follows that

$$x_5^3, \ldots, x_r^3 \in Z(G)$$

and thus linearly dependent if $r \geq 6$. Let us see that this is impossible. Arguing again by contradiction, suppose that $r \geq 6$. Without loss of generality we can suppose that

$$x_r^3 \in \langle x_5^3, \ldots, x_{r-1}^3 \rangle.$$

Using this and the fact that G is powerful we see that $[G, G]$ is generated by $x_1^3, x_2^3, \ldots x_{r-1}^3$ which implies in particular that the subgroup $\langle x_1, \ldots, x_{r-1} \rangle$ is powerful. Again using minimality of G we must then have that $\langle x_1, \ldots, x_{r-1} \rangle$ is nilpotent of class at most 2 which gives again the contradiction that $[x_1, x_2, x_3] = 1$. Hence $r \leq 5$. Finally the relations come from (B). $\Box$

So we see that minimal examples of type I are either of rank 4 or 5. In the remainder of this paper we classify these.

3.2 The 5-generator groups

From Proposition 3.3 we already have some detailed information about the structure. In particular we know that $[x_5, x_1], [x_5, x_2], [x_5, x_3], [x_5, x_4]$ are of order 3 and in the center of G. But we know that the center must be cyclic generated by $[x_1, x_2, x_3] = [x_2, x_3, x_1] = [x_3, x_1, x_2] = [x_4, x_3]^3$. Hence we get equations of the form

$$\begin{aligned}
[x_5, x_1] &= [x_2, x_3, x_1]^{\alpha_1} \\
[x_5, x_2] &= [x_3, x_1, x_2]^{\alpha_2} \\
[x_5, x_3] &= [x_1, x_2, x_3]^{\alpha_3} \\
[x_5, x_4] &= [x_3, x_4]^{3\alpha_4}
\end{aligned}$$

for some integers $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. It follows from this and the relations from Proposition 3.3 that

$$x_5[x_2, x_3]^{-\alpha_1}[x_3, x_1]^{-\alpha_2}[x_1, x_2]^{-\alpha_3} x_3^{-3\alpha_4} \in Z(G).$$

By replacing x_5 by this element we can thus assume that $x_5 \in Z(G)$. Notice that the relations of Propositions 3.3 are not affected by this.

As the center of G is cyclic, either $x_5 \in \langle z \rangle$ or $z \in \langle x_5^3 \rangle$. But G has rank 5, so we can't have $x_5 \in \langle z \rangle \leq \langle x_1, x_2, x_3, x_4 \rangle$. Hence $z \in \langle x_5^3 \rangle$. In fact we must have $z \in \langle x_5^9 \rangle$, since otherwise it follows from Proposition 3.3 that

$$x_5^3 = [x_2, x_1]^{3\alpha} \Rightarrow (x_5[x_2, x_1]^{-\alpha})^3 = 1$$

for some integer α. But then $\langle x_1, x_2, x_3, x_4 \rangle$ would be powerful, contradicting the minimality of G. So $z \in \langle x_5^9 \rangle$ which implies in particular that the order of x_5 is divisible by 3^{m+2}.

Next consider the elements x_3^9, x_4^9. Again it follows from Proposition 3.3 that these are in $Z(G)$. Now again, as the rank of G is 5 we can't have $x_5 \in \langle x_3^9 \rangle$ or $\langle x_4^9 \rangle$. Hence we must have $x_3^9, x_4^9 \in \langle x_5^3 \rangle$ and if we don't have that $x_3^9, x_4^9 \in \langle x_5^9 \rangle$ then either $x_5^3 = x_3^{9\alpha}$ or $x_5^3 = x_4^{9\alpha}$ for some integer α. But this would give $(x_5 x_3^{-3\alpha})^3 = 1$ or $(x_5 x_4^{-3\alpha})^3 = 1$ and $\langle x_1, x_2, x_3, x_4 \rangle$ would be powerful contradicting the minimality of G. Hence

$$x_3^9, x_4^9 \in \langle x_5^9 \rangle.$$

Suppose that $x_3^9 = x_5^{9\alpha}$ and $x_4^9 = x_5^{9\beta}$. Then $(x_3 x_5^{-\alpha})^9 = (x_4 x_5^{-\beta})^9 = 1$. Replacing x_3, x_4 by $x_3 x_5^{-\alpha}, x_4 x_5^{-\beta}$ we see that the relations of Proposition 3.3 are not affected. We can thus assume that $x_3^9 = x_4^9 = 1$. As $[x_4, x_3]^3 \neq 1$, we thus have

$$o(x_3) = o(x_4) = 9. \tag{13}$$

From Proposition 3.3 it follows that $[x_4, x_1], [x_4, x_2]$ are in $Z(G)$ and $[x_4, x_1]^3 = [x_4, x_2]^3 = 1$. Hence there exist some integers α_1, α_2 such that

$$\begin{aligned}
[x_4, x_1] &= [x_2, x_3, x_1]^{\alpha_1} \\
[x_4, x_2] &= [x_3, x_1, x_2]^{\alpha_2}.
\end{aligned}$$

It follows that $x_4[x_2, x_3]^{-\alpha_1}[x_3, x_1]^{-\alpha_2}$ commutes with x_1 and x_2. Replacing x_4 by this element does not affect the relations of Proposition 3.3 nor does it affect (13). So we can assume that

$$[x_4, x_1] = [x_4, x_2] = 1. \tag{14}$$

We have already seen that the order of x_5 is divisible by 3^{m+2}. Replacing x_1 by x_1x_5 if necessary, we can then assume that x_1 has order divisible by 3^{m+2}. As the relations from Proposition 3.3 are symmetrical in x_1, x_2 (swapping x_1 and x_2 and replacing z, x_4 by z^{-1}, x_4^{-1} gives us the same relations), we can assume that $o(x_1) \geq o(x_2)$. Since $x_2^{3^{m+1}}, x_1^{3^{m+1}} \in Z(G)$ as one sees from Proposition 3.3, it follows then that $x_2^{3^{m+1}} = x_1^{3^{m+1}\alpha}$, for some integer α, and $(x_2x_1^{-\alpha})^{3^{m+1}} = 1$. Again the relations of Proposition 3.3 are not affected if we replace x_2 by $x_2x_1^{-\alpha}$. As $[x_2, x_1]^{3^m} = z^{3^{m-1}} \neq 1$, we can thus assume that

$$o(x_2) = 3^{m+1}. \tag{15}$$

We have seen previously that we can assume that $o(x_1)$ is divisible by 3^{m+2}. So $x_1^{3^{m+1}}$ is a non-trivial element in the center of G. Using the minimality of G as we have done previously we see that we can't have $x_5^3 \in \langle x_1^{3^{m+1}} \rangle$ and as the center is cyclic we must have

$$x_1^{3^{m+1}} \in \langle x_5^9 \rangle. \tag{16}$$

We now turn to the commutator relations. As G is powerful we have a relation of the form

$$[x_2, x_1] = x_1^{3\alpha_1} x_2^{3\alpha_2} x_3^{3\alpha_3} x_4^{3\alpha_4} x_5^{3\alpha_5}.$$

Taking commutator on both sides with x_4 and using the relations from Proposition 3.3, we see that $1 = z^{-3^{m-1}\alpha_3}$. As $o(z) = 3^m$ it follows that $3|\alpha_3$. Taking commutators with x_1 and x_2 shows similarly that α_1 and α_2 are divisible by 3^m. It now follows from (13)-(16) that

$$[x_2, x_1] = x_4^{3\alpha_4} x_5^{3\alpha_5}$$

for some integers α_4, α_5. As $[x_2, x_1, x_3] = z^{-3^{m-1}}$ we see that we can take $\alpha_4 = -1$ (remember that $o(x_4) = 9$). So we have

$$[x_2, x_1] = x_4^{-3} x_5^{3\alpha} \tag{17}$$

for some integer α. Next consider

$$[x_3, x_2] = x_1^{3\alpha_1} x_2^{3\alpha_2} x_3^{3\alpha_3} x_4^{3\alpha_4} x_5^{3\alpha_5}$$

Arguing in a similar way as we did before, taking commutators with x_4, x_3, x_2, x_1, shows that we get an equation of the form

$$[x_3, x_2] = x_2^{-3^m} x_5^{3\alpha_5}.$$

(Notice that $[x_3, x_2, x_1] = z^{-3^{m-1}}$, $o(x_2) = 3^{m+1}$ and that $[x_2, x_1]^{3^m} = z^{3^{m-1}}$). Taking the third power on both sides gives

$$1 = x_2^{-3^{m+1}} x_5^{9\alpha_5} = x_5^{9\alpha_5}$$

and as we had established before that $o(x_5)$ is divisible by 3^{m+2}, we can deduce that $\alpha_5 = 3^m l$ for some integer l. Thus

$$[x_3, x_2] = x_2^{-3^m} x_5^{3^{m+1} l} = (x_2 x_5^{-3l})^{-3^m}.$$

Replacing x_2 by $x_2 x_5^{-3l}$ does not change the relations in Proposition 3.3 nor does it change the relations (13)-(17). (Notice that $o((x_2 x_5^{-3l})^{3^m}) = o([x_3, x_2]) = 3$ and thus $o(x_2 x_5^{-3l}) = o(x_2)$). So we can assume that

$$[x_3, x_2] = x_2^{-3^m}. \tag{18}$$

Next we analyse

$$[x_4, x_3] = x_1^{3\alpha_1} x_2^{3\alpha_2} x_3^{3\alpha_3} x_4^{3\alpha_4} x_5^{3\alpha_5}.$$

Taking commutators with x_3, x_4, x_1, x_2 and using Proposition 3.3 and (13)-(18) shows that we can replace this by an equation of the form

$$[x_4, x_3] = x_5^{3\alpha_5}.$$

This means that $[x_4, x_3]$ is an element of $\langle x_5^3 \rangle$ of order 9 and as $o(x_5)$ is divisible by 3^{m+2} we get an equation of the form

$$[x_4, x_3] = x_5^{3^m \gamma} \tag{19}$$

for some integer γ. We are now only left with the commutator

$$[x_3, x_1] = x_1^{3\alpha_1} x_2^{3\alpha_2} x_3^{3\alpha_3} x_4^{3\alpha_4} x_5^{3\alpha_5}.$$

Taking commutators with x_3, x_4, x_1, x_2 and using previous relations gives an equation of the form

$$[x_3, x_1] = x_1^{-3^m} x_5^{3\beta} \tag{20}$$

for some integer β (notice that $x_1^{3^{m+1}} \in \langle x_5^9 \rangle$).

We have seen earlier that $o(x_5)$ is divisible by 3^{m+2}. Suppose that $o(x_5) = 3^{m+2+i}$ where i is some non-negative integer. From the relations of Proposition 3.3 we know that $o([x_2, x_1]) = 3^{m+1}$ which implies that α in (17) is of the form $3^i \cdot \bar{\alpha}$ where $\bar{\alpha}$ is not divisible by 3. From Proposition 3.3 we also know that $o([x_4, x_3]) = 9$ and thus γ from (19) is of the form $\gamma = 3^i \bar{\gamma}$ where $\bar{\gamma}$ is not divisible by 3. Let us summarize the commutator relations. These as we have seen can be written of the form

$$\begin{array}{rcl} [x_2, x_1] & = & x_4^{-3} x_5^{3^{i+1}\alpha} \\ [x_3, x_1] & = & x_1^{-3^m} x_5^{3\beta} \\ [x_3, x_2] & = & x_2^{-3^m} \\ [x_4, x_1] & = & 1 \\ [x_4, x_2] & = & 1 \\ [x_4, x_3] & = & x_5^{3^{m+i}\gamma} \end{array} \qquad \text{(C)}$$

where $o(x_5) = 3^{m+2+i}$ and $m \geq 2$. Let us analyse these more closely. Firstly notice that β is not divisible by 3 since otherwise the commutator relations show that

$$\langle x_1, x_2, x_3, x_4^{-1} x_5^{3^i \cdot \alpha}, x_5^3 \rangle$$

is powerful. Hence none of α, β, γ is divisible by 3. Replacing x_5 by x_5^{α} we can assume that $\alpha = 1$. It follows in particular that

$$z^{3^{m-1}} = [x_2, x_1]^{3^m} = x_5^{3^{m+i+1}}.$$

From Proposition 3.3 and (C) we then also have

$$x_5^{3^{m+i+1}} = z^{3^{m-1}} = [x_4, x_3]^3 = x_5^{3^{m+i+1}\gamma}.$$

We can take γ to be an integer between 0 and 9 and by this equation γ is a unit modulo 9 (being 1 modulo 3). Now let τ be the inverse of γ modulo 9. Replacing x_4 by x_4^{τ} we can assume that $\gamma = 1$. Notice that the other equations of (C) are unaffected and the same is true for Proposition 3.3 and (13), (15)-(16). Having seen that we can take α, γ to be 1 we turn to β. Now x_5^3 has order 3^{m+1+i} and as β is not divisible by 3 it is a unit modulo 3^{m+i+1}.

Let σ be the inverse of β modulo 3^{m+i+1}. Replace x_1 by x_1^{σ} and x_2 by x_2^{β}. With this change we can assume that β in (C) is 1 and it is easy to check that all the other relations remain the same. Notice that the second relation of (C) together with the established fact that $[x_3, x_1]^3 = 1$ (Proposition 3.3) implies that

$$x_1^{3^{m+1}} = x_5^9. \tag{21}$$

One can easily check that all the relations in Proposition 3.3 remain the same after the change of x_1, x_2 and x_4 and the same is true for the power relations. As the relations in Proposition 3.3 are consequences of (C) they remain the same and the same is true for the power relations (13) and (15). We have thus arrived at a two parameter family of candidates for minimal examples of type I generated by 5 elements. These are the groups $A(i,m)$ where $A(i,m) = \langle x_1, x_2, x_3, x_4, x_5 \rangle$ is the largest nilpotent group of class at most 3 satisfying the extra relations

$$A(i,m) \qquad \begin{aligned} &[x_2, x_1] = x_4^{-3} x_5^{3^{i+1}} \\ &[x_3, x_1] = x_1^{-3^m} x_5^3 \\ &[x_3, x_2] = x_2^{-3^m} \\ &[x_4, x_1] = 1 \\ &[x_4, x_2] = 1 \\ &[x_4, x_3] = x_5^{3^{m+i}} \\ &[x_5, x_j] = 1, \quad 1 \le j \le 4 \\ &x_1^{3^{m+1}} = x_5^9 \\ &x_2^{3^{m+1}} = 1 \\ &x_3^9 = 1 \\ &x_4^9 = 1 \\ &x_5^{3^{m+i+2}} = 1. \end{aligned}$$

Theorem 3.4 *The minimal examples of type I that are of rank 5 consist of the two parameter family $A(i,m)$, $i \ge 0$, $m \ge 2$. The group $A(i,m)$ has class 3, exponent 3^{2m+i+1} and order 3^{3m+i+8}. Furthermore the members of the family are pairwise non-isomorphic.*

Proof. Let $F(i,m)$ be the largest group satisfying all the relations defining $A(i,m)$ except the relation $x_1^{3^{m+1}} = x_5^9$. It is easy to see that it follows from the relations that every element in $F(i,m)$ can be written of the form

$$x_1^{n_1}x_2^{n_2}x_3^{n_3}x_4^{n_4}x_5^{n_5}$$

with $0 \le n_1 < 3^{2m+i+1},\ 0 \le n_2 < 3^{m+1},\ 0 \le n_3, n_4 < 9,\ 0 \le n_5 < 3^{m+i+2}$. It follows that $F(i,m)$ has order at most $3^{4m+2i+8}$. We first show that this is the exact order of $F(i,m)$. In order to show this we consider the set of all formal expressions

$$a_1^{n_1}a_2^{n_2}a_3^{n_3}a_4^{n_4}a_5^{n_5}$$

where $0 < n_1 < 3^{2m+i+1},\ 0 \le n_2 < 3^{m+1},\ 0 \le n_3, n_4 < 9,\ 0 \le n_5 < 3^{m+i+2}$. We define a product on these formal expressions (a formula derived from the relations of $F(i,m)$) by setting

$$\begin{aligned} a_1^{n_1}a_2^{n_2}a_3^{n_3}a_4^{n_4}a_5^{n_5} * a_1^{m_1}a_2^{m_2}a_3^{m_3}a_4^{m_4}a_5^{m_5} \quad = \quad & a_1^{n_1+m_1-3^m n_3 m_1} a_2^{n_2+m_2-3^m n_3 m_2} \\ & a_3^{n_3+m_3} a_4^{n_4+m_4-3n_2m_1} \\ & a_5^{n_5+m_5+3n_3m_1+3^{i+1}n_2m_1+3^{m+i}n_4m_3} \\ & a_5^{3^{m+i+1}(n_3n_2m_1-m_3m_1n_2)} \end{aligned}$$

and where the exponents of a_1, a_2, a_3, a_4, a_5 are calculated modulo $3^{2m+i+1}, 3^{m+1}$, $9, 9, 3^{m+i+2}$. Straightforward calculations show that we get a group which satisfies all the relations of $F(i,m)$. Now let $N = \langle x_1^{3^{m+1}} x_5^{-9} \rangle$. One checks easily that N is a normal subgroup. Now $G = A(i,m) = F(i,m)/N$ and has exactly $n(G) = 3^{4m+2i+8}/3^{m+i} = 3^{3m+i+8}$ elements. In particular x_5 has order 3^{m+i+2}. As $G^{3^{2m+i}} = \langle x_5^{3^{m+i+1}} \rangle$ and $[a_1, a_2, a_3] = x_5^{3^{m+i+1}}$, it follows that the class is 3 and the exponent is $e(G) = 3^{2m+i+1}$. As $3^{m+7} = n(G)/e(G)$, we have that m and i are determined by the exponent and the order of G. Hence the map $(i,m) \mapsto (n(G), m(G))$ is injective and the groups in the list are thus pairwise non-isomorphic.

It remains to establish the minimality of $A(i,m)$. Let H/K be a section of $G = A(i,m)$ that is powerful and of class 3. As $\gamma_3(H) \neq \{1\}$ and G is powerful, we must have elements $g_1, g_2, g_3 \in H$ of the following form

$$\begin{aligned} g_1 &= x_1 x_4^{r_1} x_5^{s_1} \\ g_2 &= x_2 x_4^{r_2} x_5^{s_2} \\ g_3 &= x_3 x_4^{r_3} x_5^{s_3}. \end{aligned}$$

Now for H/K to remain of class 3, we can't have $x_5^{3^{m+1+i}} \in K$. Let us see what restrictions this makes on K. Suppose

$$g = x_1^{n_1} x_2^{n_2} x_3^{n_3} x_4^{n_4} x_5^{n_5}$$

is in K. Firstly as $[g, g_1, g_2] = x_5^{3^{m+1+i} n_3}$ is in K, we must have $3|n_3$. We use in the following the fact that $[x_i, x_1]^3 = [x_i, x_2]^3 = 1$ for $i = 3, 4, 5$ and that $[x_4, x_3]^9 = 1$. As $[g, g_1]^3$, $[g, g_2]^3 \in K$, we get that $[x_2, x_1]^{3n_1} = x_5^{3^{i+2} n_1}$ and $[x_2, x_1]^{3n_2} = x_5^{3^{i+2} n_2}$ are in K which implies that 3^m divides n_1 and n_2. Next as $[g, g_3] = [x_4, x_3]^{n_4} = x_5^{3^{m+i} n_4} \in K$, we have that $9|n_4$. So g is of the form

$$g = x_1^{3^m r_1} x_2^{3^m r_2} x_3^{3 r_3} x_5^{r_5}.$$

Notice that there is no occurrence of x_4 in g. We next calculate modulo K and use the fact that H/K is powerful. As $[\bar{g_1}, \bar{g_2}] = \bar{x_4}^{-3} \bar{x_5}^{3^{i+1}}$ is not in $\langle \bar{g_1}^3, \bar{g_2}^3, \bar{g_3}^3, \bar{x_5}^3 \rangle$, we must have that H contains an element g_4 of the form

$$g_4 = x_4 x_5^{s_5}.$$

Replacing g_i by a suitable $g_i g_4^{t_i}$, $i = 1, 2, 3$, we can now assume that

$$\begin{aligned} g_1 &= x_1 x_5^{s_1} \\ g_2 &= x_2 x_5^{s_2} \\ g_3 &= x_3 x_5^{s_3} \\ g_4 &= x_4 x_5^{s_4}. \end{aligned}$$

Now consider $[g, g_1], [g, g_2] \in K$. We conclude that $x_5^{3^{m+i+1} r_1}, x_5^{3^{m+i+1} r_2} \in K$ and hence both r_1, r_2 are divisible by 3 and it follows that g is of the form

$$g = x_3^{3 r_3} x_5^{r_5}.$$

Now $[g, g_4] \in K$ gives us that $x_5^{3^{m+i+1} r_3} \in K$ which forces r_3 to be divisible by 3. Hence g must be a power of x_5. But as K cannot contain $x_5^{3^{m+1+i}}$, it follows that g must then be trivial. Hence $K = \{1\}$. Now as H must be powerful and $[g_3, g_1] = x_1^{-3^m} x_5^3 \in H$ is not in $\langle g_1^3, g_2^3, g_3^3, g_4^3 \rangle$, we must have that H contains x_5. It is now clear that $H = G$. This finishes the proof. $\square$

3.3 The 4-generator groups

As for the 5-generator groups the following group is going to help in clarifying the structure of the minimal examples.

$$R = \{x \in G \,:\, [x,g]^3 \in \gamma_3(G)\ \forall g \in G\}.$$

Then it follows from Proposition 3.3 that

$$R = \langle x_1^{3^{m-1}}, x_2^{3^{m-1}}, x_3, x_4\rangle$$

and

$$R \cap Z_2(G) = \langle x_1^{3^{m-1}}, x_2^{3^{m-1}}, x_3^3, x_4\rangle,$$

where $Z_2(G)$ is the second center of G. From Proposition 3.3 we know that $x_1^{3^m}, x_2^{3^m}, x_3^3, x_4^3 \not\in Z(G)$ but $x_1^{3^{m+1}}, x_2^{3^{m+1}}, x_3^9$, x_4^9 are all elements in the center. Without loss of generality we can assume that $o(x_2)|o(x_1)$. Suppose that $x_2^{3^{m+1}} = x_1^{3^{m+1}e}$. Then replacing x_2 by $x_2x_1^{-e}$ we can assume that

$$o(x_2) = 3^{m+1},\ 3^{m+1}|o(x_1). \tag{22}$$

Without loss of generality (replacing x_1 by $x_1x_3^{\alpha}x_4^{\beta}$) we can assume that $o(x_1)$ is not less than the order of x_3 or x_4. By replacing then x_4 by suitable $x_4x_1^{3^m f}$ we can assume that $o(x_4) \geq o(x_1^{3^m})$. By replacing x_3 by some $x_3x_4^e$ we can assume that $o(x_3) \geq o(x_4)$. Then by replacing x_4 by some suitable $x_4x_3^{3f}$ we can assume further that

$$o(x_3) \geq o(x_4) \geq o(x_3^3). \tag{23}$$

Note that these changes have no effect on relations given by Proposition 3.3.

Lemma 3.5 $o(x_4) \geq o(x_1^{3^{m-1}})$.

Proof Otherwise $o(x_4) \leq o(x_1^{3^m})$ and as $o(x_3) \geq o(x_4) \geq o(x_3^3)$ we have that $o(x_3) \leq o(x_1^{3^{m-1}})$. It follows that

$$\begin{aligned} x_3^9 &= x_1^{3^{m+1}\alpha} \\ x_4^9 &= x_1^{3^{m+2}\beta} \end{aligned}$$

for some integers α, β. By replacing x_3 by $x_3x_1^{-3^{m-1}\alpha}$ and x_4 by $x_4x_1^{-3^m\beta}$, we can assume that

$$x_3^9 = x_4^9 = 1.$$

Notice that we may loose the property that $[x_3, x_2]^3 = 1$ but we still have $[x_3, x_2]^3$ is in $Z(G)$ of order at most 3. Now as G is powerful we get an equation of the following form

$$[x_3, x_1] = x_1^{3\alpha_1} x_2^{3\alpha_2} x_3^{3\alpha_3} x_4^{3\alpha_4}.$$

Since $[x_3, x_1]$ commutes with x_1^3, x_2^3 we must have that α_1, α_2 are both divisible by 3. Moreover by Proposition 3.3, $[x_3, x_1]$ commutes with x_4, x_3, x_1, and thus we must have that $3|\alpha_3$, $3|\alpha_4$ and $3^m|\alpha_2$. As furthermore $[x_3, x_1, x_2] = z^{3^{m-1}}$ it follows that

$$[x_3, x_1] = x_1^{3^m \alpha}, \quad \alpha \text{ is not divisible by 3}$$

and as $[x_3, x_1]^3 = 1$ it follows that

$$o(x_1) = 3^{m+1}.$$

Then consider

$$[x_2, x_1] = x_1^{3\beta_1} x_2^{3\beta_2} x_3^{3\beta_3} x_4^{3\beta_4}.$$

As this element commutes with x_4, x_2 and x_1 we see that

$$[x_2, x_1] = x_4^{3\beta_4}$$

and $[x_2, x_1]^3 = 1$ which contradicts Proposition 3.3. $\Box$

If H is any subgroup of G, we will use the notation $e(H)$ for the exponent of H. It follows from the last lemma that

$$\begin{aligned} e(R) &= o(x_3) \\ e(R \cap Z^2(G)) &= o(x_4). \end{aligned}$$

Notice however that this is just the case for the time being and this will not remain so. We will deal with two cases, each of which will lead to a family of minimal examples.

3.3.1 Case 1. $e(R) = e(R \cap Z^2(G))$

Here $o(x_3) = o(x_4)$ and it follows that $x_3^9 = x_4^{9\alpha}$ for some integer α. Replacing x_3 by $x_3 x_4^{-\alpha}$ we can assume that

$$x_3^9 = 1. \tag{24}$$

We claim that in this case we must have $o(x_4) > o(x_1^{3^{m-1}})$. Otherwise $o(x_4) \leq o(x_1^{3^{m-1}})$ and $x_4^9 = x_1^{3^{m+1}\alpha}$ for some integer α. By replacing x_4 by $x_4 x_1^{-3^{m-1}\alpha}$, we can assume that

$$x_4^9 = 1.$$

Now similar argument as in Lemma 3.6 gives us a contradiction.

We next deduce what the order of x_4 is. As G is powerful and as $[x_2, x_1]$ commutes with x_1, x_2 and x_4, we have that

$$[x_2, x_1] = x_4^{3\alpha} x_1^{3^{m+1}\beta}$$

for some integers α and β. And as $[x_2, x_1]$ does not commute with x_3, α is not divisible by 3. As we have seen previously $o(x_4) > o(x_1^{3^{m-1}})$ and it follows that $o(x_4^3) = o([x_2, x_1]) = 3^{m+1}$ and therefore

$$o(x_4) = 3^{m+2}. \tag{25}$$

Also, as $o(x_4) > o(x_1^{3^{m-1}})$,

$$x_1^{3^{m+1}} = x_4^{3^{2+i}\tau}, \quad 1 \leq i \leq m-1 \tag{26}$$

where τ is not divisible by 3. (Notice that $o(x_1) \geq o(x_4)$ implies that $2 + i \leq m + 1$). It follows in particular that $o(x_1) = 3^{2m+1-i}$. We need now to sort out the commutator relations. Now $[x_4, x_1], [x_4, x_2]$ are elements in the center of order dividing 3. Replacing x_4 by some $x_4 x_1^{3^m e} x_2^{3^m f}$ we can assume that

$$\begin{aligned} [x_4, x_1] &= 1 \\ [x_4, x_2] &= 1 \\ [x_4, x_3] &= x_4^{3^m\delta} \end{aligned} \tag{27}$$

where δ is not divisible by 3. (Notice that $x_2^{3^{m+1}} = x_3^9 = 1$ and $x_1^{3^{m+1}} \in \langle x_4^9 \rangle$ so $Z(G) = \langle x_4^9 \rangle$). These relations together with Proposition 3.3 also imply that

$$[x_2, x_1] = x_4^{-3(1+3\alpha)} \tag{28}$$

for some integer α. Next using the fact that $[x_3, x_1]$ commutes with x_1, x_3, x_4, $[x_3, x_1, x_2] = z^{3^{m-1}}$ and $[x_3, x_1]^3 = 1$, it follows from (26) that

$$[x_3, x_1] = x_1^{-3^m} x_4^{3^{1+i}(\tau + 3^{m-i}\beta)} \tag{29}$$

for some integer β. Then using the fact that $[x_3, x_2]$ is of order 3, commuting with x_2, x_3, x_4 and observing that $[x_3, x_2, x_1] = z^{-3^{m-1}}$, we see that

$$[x_3, x_2] = x_2^{-3^m} x_4^{3^{m+1}\gamma} \tag{30}$$

for some integer γ. We make some small changes in order to reach a unique presentation (in terms of m and i). First by replacing x_2 by $x_2 x_4^{-3\gamma}$ we can assume that γ in (30) is zero. Next we use the relations just obtained together with Proposition 3.3, to observe that

$$x_4^{3^{m+1}\delta} = [x_4, x_3]^3 = [x_2, x_1]^{3^m} = x_4^{-3^{m+1}}$$

to deduce that we can assume the $\delta = 1 + 3\epsilon$ for some integer ϵ. Now let μ be the inverse of $\tau + 3^{m-i}\beta$ modulo 3^{m+1} and replace x_1 by x_1^{μ} and x_2 by $x_2^{\tau+3^{m-i}\beta}$. This results in the same relations as before except that in (29) we have $\tau = 1$ and $\beta = 0$. Finally let $1 + 3r$ be the inverse of $1 + 3\alpha$ modulo 3^m and let $1 + 3s$ be the inverse of $1 + 3\epsilon$ modulo 9. Replacing x_2 and x_3 by x_2^{1+3r} and x_3^{1+3s} we can furthermore assume that $\delta = 1$ and $\alpha = 0$ in (27) and (28). We thus arrive at the following unique candidate:

$$B(i,m) \qquad \begin{aligned}
&[x_2, x_1] = x_4^{-3}\\
&[x_3, x_1] = x_1^{-3^m} x_4^{3^{1+i}}\\
&[x_3, x_2] = x_2^{-3^m}\\
&[x_4, x_1] = 1\\
&[x_4, x_2] = 1\\
&[x_4, x_3] = x_4^{3^m}\\
&x_1^{3^{m+1}} = x_4^{3^{2+i}}\\
&x_2^{3^{m+1}} = 1\\
&x_3^{9} = 1\\
&x_4^{3^{2+m}} = 1.
\end{aligned}$$

Here $m \geq 2$ and $1 \leq i \leq m - 1$.

Theorem 3.6 *The minimal examples of type I that are of rank 4 and have the extra property that* $e(R) = e(R \cap Z^2(G))$ *consist of the two parameter family* $B(i,m)$, $1 \leq i \leq m-1$, $m \geq 2$. *The group* $B(i,m)$ *has class 3, exponent* 3^{2m-i+1} *and order* 3^{3m+6}. *Furthermore the members of the family are pairwise non-isomorphic.*

Proof We argue in a similar manner as in the proof of Theorem 3.4. Let $F(i,m)$ be the largest group satisfying all the relations defining $A(i,m)$ except the relation $x_1^{3^{m+1}} = x_4^{3^{2+i}}$. As $F(i,m)$ is powerful every element in $F(i,m)$ can be written of the form

$$x_1^{n_1}x_2^{n_2}x_3^{n_3}x_4^{n_4}$$

with $0 \leq n_1 < 3^{2m-i+1}$, $0 \leq n_2 < 3^{m+1}$, $0 \leq n_3 < 9$, $0 \leq n_4 < 3^{m+2}$. It follows that $F(i,m)$ has at most 3^{4m-i+6} elements. We first show that this is the exact order of $F(i,m)$. In order to show this we consider the set of all formal expressions

$$a_1^{n_1}a_2^{n_2}a_3^{n_3}a_4^{n_4}$$

where $0 < n_1 < 3^{2m-i+1}$, $0 \leq n_2 < 3^{m+1}$, $0 \leq n_3 < 9$, $0 \leq n_4 < 3^{m+2}$. We define a product on these formal expressions (a formula derived from the relations of $F(i,m)$) by setting

$$\begin{aligned} a_1^{n_1}a_2^{n_2}a_3^{n_3}a_4^{n_4} * a_1^{m_1}a_2^{m_2}a_3^{m_3}a_4^{m_4} \quad = \quad & a_1^{n_1+m_1-3^m n_3 m_1} a_2^{n_2+m_2-3^m n_3 m_2} \\ & a_3^{n_3+m_3} \\ & a_4^{n_4+m_4-3n_2m_1+3^{i+1}n_3m_1-3^m n_4 m_3} \\ & a_4^{3^{m+1}(n_2m_1m_2-n_2n_3m_1)} \end{aligned}$$

and where the exponents of a_1, a_2, a_3, a_4 are calculated modulo 3^{2m-i+1}, 3^{m+1}, $9, 3^{m+2}$. Straightforward calculations show that we get a group which satisfies all the relations of $F(i,m)$. Now let $N = \langle x_1^{3^{m+1}} x_4^{-3^{2+i}} \rangle$. One checks easily that N is a normal subgroup. Now $B(i,m) = F(i,m)/N$ and has exactly $n(G) = 3^{4m-i+6}/3^{m-i} = 3^{3m+6}$ elements. In particular x_1 has order 3^{2m-i+1}. As $G^{3^{2m-i}} = \langle x_1^{3^{2m-i}}, x_4^{3^{2m-i}} \rangle = \langle x_4^{3^{m+1}} \rangle$ and $[x_1, x_2, x_3] = x_4^{-3^{m+1}}$, it follows that the class is 3 and the exponent is $e(G) = 3^{2m-i+1}$. Now m and i are determined by the exponent and the order of G. Hence the map $(i,m) \mapsto (n(G), m(G))$ is injective and the groups in the list are thus pairwise non-isomorphic.

It remains to establish the minimality of $B(i,m)$. Let H/K be a section of $G = B(i,m)$ that is powerful and of class 3. As $\gamma_3(H) \neq \{1\}$ and G is powerful, we must have elements $g_1, g_2, g_3 \in H$ of the following form

$$g_1 \quad = \quad x_1 x_4^{r_1}$$

$$\begin{aligned} g_2 &= x_2 x_4^{r_2} \\ g_3 &= x_3 x_4^{r_3}. \end{aligned}$$

Now for H/K to remain of class 3, we can't have $x_4^{3^{m+1}} \in K$. Let us see what restrictions this makes on K. Suppose

$$g = x_1^{n_1} x_2^{n_2} x_3^{n_3} x_4^{n_4}$$

is in K. Firstly as $[g, g_1, g_2] = x_4^{3^{m+1} n_3}$ is in K, we must have $3|n_3$. We use in the following the fact that $[x_i, x_1]^3 = [x_i, x_2]^3 = 1$ for $i = 3, 4$ and that $[x_4, x_3]^9 = 1$. As $[g, g_1]^3$, $[g, g_2]^3 \in K$, we get that $[x_2, x_1]^{3n_1} = x_4^{-3^2 n_1}$ and $[x_2, x_1]^{3n_2} = x_4^{-3^2 n_2}$ are in K which implies that 3^m divides n_1 and n_2. Next as $[g, g_3] = [x_4, x_3]^{n_4} = x_4^{-3^m n_4} \in K$, we have that $9|n_4$. So g is of the form

$$g = x_1^{3^m r_1} x_2^{3^m r_2} x_3^{3 r_3} x_4^{9 r_4}.$$

Using the fact that the commutators of g with g_1, g_2 are in K we see that both r_1 and r_2 are divisible by 3 and it follows using the relations of G that g can be written of the form $g = x_3^{3\alpha} x_4^{9\beta}$. As $g^3 \in K$ we must have that β is divisible by 3^{m-1}. Hence g is of the form

$$g = x_3^{3\alpha} x_4^{3^{m+1}\gamma}.$$

If for all such $g \in K$, α is divisible by 3, then all $g \in K$ must be trivial as $x_4^{3^{m+1}}$ is not in K. Hence $K = \{1\}$ and H is powerful. But $[g_1, g_2] = x_4^3$ is not in $\langle g_1^3, g_2^3, g_3^3, x_4^9 \rangle$ and H therefore has to contain x_4 to be powerful. Hence $H = G$ and we are done. So we can assume that 3 does not divide α for some g. In order to avoid K containing $x_4^{3^{m+1}}$, K must then be cyclic generated by some $g = x_3^{3\alpha} x_4^{3^{m+1}\gamma}$. Now $K = \langle g \rangle = \langle g_3 x_4^t \rangle$ for some integer t. Using the fact that $[g_3 x_4^t, g_3] \in K$, we see that t must be divisible by 9. We want to show that H must contain x_4. If this was not the case then H would be a subgroup of $\langle g_1, g_2, g_3, x_4^3 \rangle$ and thus modulo $K = \langle g_3 x_4^t \rangle$ we would have

$$[g_1, g_2] = x_4^3$$

must be in $\langle g_1^3, g_2^3, g_3^3, x_4^9 \rangle$ and hence x_4^3 would be in $\langle g_1^3, g_2^3, g_3^3, x_4^9 \rangle$. But this is not the case. It follows that $x_4 \in H$ and $G = H$. In particular $[g, x_4] = x_4^{3^{m+1}\alpha} \in K$ and hence 3 divides α contrary to our assumption. This finishes the proof. $\square$

3.3.2 Case 2. $e(R) > e(R \cap Z^2(G))$

Here $o(x_3) > o(x_4)$ and $x_4^9 = x_3^{27\alpha}$ for some integer α. By replacing x_4 by $x_4 x_3^{-3\alpha}$ we can assume that

$$x_4^9 = 1. \tag{31}$$

We now go on to clarify the structure further. Notice first that

$$x_1^{3^{m+1}} = x_3^{3^{2+i}\tau}, \quad 3 \not| \tau, \quad 1 \leq i \leq m-1. \tag{32}$$

This follows from the fact that $o(x_1) \geq o(x_3) > o(x_1)^{3^{m-1}}$. As $[x_3, x_1]$ commutes with x_1, x_3, x_4, $[x_3, x_1]^3 = 1$ and $[x_3, x_1, x_2] = [x_2, x_1]^{3^m}$, it follows from the last equation that

$$[x_3, x_1] = x_1^{-3^m} x_3^{3^{1+i}(\tau+3\beta)} \tag{33}$$

for some integer β. Using these equations and the fact that $[x_2, x_1]$ commutes with x_1, x_2, x_4 and $[x_2, x_1, x_3] = [x_4, x_3]^{-3}$, we see that

$$[x_2, x_1] = x_4^{-3} x_3^{3^{2+j}\gamma} \tag{34}$$

where $j \geq 0$ and $3 \not| \gamma$. As $o([x_2, x_1]) = 3^{m+1}$ we get in particular

$$o(x_3) = 3^{3+j+m}. \tag{35}$$

Notice that it follows then from (32) that

$$o(x_1) = 3^{2m-i+j+2}.$$

We next turn our attention to $[x_3, x_2]$. Using similar arguments as before we see that

$$[x_3, x_2] = x_2^{-3^m} x_3^{3^{2+j+m}\epsilon} \tag{36}$$

and

$$[x_4, x_3] = x_3^{3^{1+j+m}\delta} \tag{37}$$

for some integers ϵ and δ where δ is not divisible by 3. With appropriate replacements we can also assume that

$$[x_4, x_1] = 1 \tag{38}$$

$$[x_4, x_2] = 1 \tag{39}$$

We now make some fine replacements in order to reach a clean presentation. First replacing x_2 by $x_2 x_3^{-3^{2+j}\epsilon}$ we can assume that ϵ in (36) is zero. Now

$$x_3^{3^{2+j+m}\delta} = [x_4, x_3]^3 = [x_2, x_1]^{3^m} = x_3^{3^{2+j+m}\gamma}$$

which implies that $\gamma = \delta$ modulo 3. Replacing x_1, x_4 by x_1^δ, x_4^δ we can assume that

$$\begin{aligned} \gamma &= 1 + 3a && (40) \\ \delta &= 1 + 3b. && (41) \end{aligned}$$

Next replacing x_1, x_2 by x_1^τ, x_2^τ. These equations are not affected and we get furthermore that $\tau + 3\beta = 1 + 3c$. We have thus arrived at

$$\begin{aligned} [x_2, x_1] &= x_4^{-3} x_3^{3^{2+j}(1+3a)} \\ [x_3, x_1] &= x_1^{-3^m} x_3^{3^{1+i}(1+3c)} \\ [x_3, x_2] &= x_2^{-3^m} \\ [x_4, x_1] &= 1 \qquad\qquad \text{(D)} \\ [x_4, x_2] &= 1 \\ [x_4, x_3] &= x_3^{3^{1+j+m}(1+3b)}. \end{aligned}$$

Let $1 + 3e$ be the inverse of $1 + 3b$ modulo 9 and $1 + 3f$ be the inverse of $1 + 3c$ modulo 3^{m+1+j}. Replacing x_4, x_1 by x_4^{1+3e}, x_1^{1+3f} and replacing x_2 appropriately gives us the presentation

$$C(j,i,m) \qquad \begin{aligned} &[x_2, x_1] = x_4^{-3} x_3^{3^{2+j}} \\ &[x_3, x_1] = x_1^{-3^m} x_3^{3^{1+i}} \\ &[x_3, x_2] = x_2^{-3^m} \\ &[x_4, x_1] = 1 \\ &[x_4, x_2] = 1 \\ &[x_4, x_3] = x_3^{3^{1+j+m}} \\ &x_1^{3^{m+1}} = x_3^{3^{2+i}} \\ &x_2^{3^{m+1}} = 1 \\ &x_4^9 = 1 \\ &x_3^{3^{3+j+m}} = 1. \end{aligned}$$

Here $m \geq 2$,$j \geq 0$ and $1 \leq i \leq m - 1$.

Theorem 3.7 *The minimal examples of type I that are of rank 4 and have the extra property that* $e(R) > e(R \cap Z^2(G))$ *consist of the three parameter family* $C(j, i, m)$, $1 \leq i \leq m-1$, $m \geq 2, j \geq 0$. *The group* $C(j, i, m)$ *has class* 3, *exponent* $3^{2m+2+j-i}$ *and order* 3^{3m+j+7}. *Furthermore the members of the family are pairwise non-isomorphic.*

Proof We argue in a similar manner as in the proof of Theorem 3.4. First let

$$(y_1, y_2, y_3, y_4) = (x_1, x_2, x_3 x_1^{-3^{m-i-1}}, x_4).$$

The defining relations for these new generators are

$$\begin{aligned}
&[y_1, y_2] = y_4^3 y_1^{-3^{m+j-i+1}} y_3^{-3^{2+j}} \\
&[y_3, y_1] = y_3^{3^{1+i}} \\
&[y_3, y_2] = y_2^{-3^m} y_4^{-3^{m-i}} y_1^{3^{2m+j-2i}+3^{3m+j-2i}} y_3^{3^{m+j-i+1}} \\
&[y_1, y_4] = 1 \\
&[y_4, y_2] = 1 \\
&[y_3, y_4] = y_1^{-3^{2m+j-i}} \\
&y_1^{3^{2m+j-i+2}} = 1 \\
&y_2^{3^{m+1}} = 1 \\
&y_3^{3^{2+i}} = 1 \\
&y_4^9 = 1.
\end{aligned}$$

As $G = C(j, i, m)$ is powerful every element can be written of the form

$$y_2^{n_2} y_4^{n_4} y_1^{n_1} y_3^{n_3}$$

with $0 \leq n_1 < 3^{2m+j-i+2}$, $0 \leq n_2 < 3^{m+1}$, $0 \leq n_3 < 3^{2+i}$, $0 \leq n_4 < 3^2$. It follows that $C(j, i, m)$ has at most 3^{3m+j+7} elements. We first show that this is the exact order of $C(j, i, m)$. In order to show this we realize the group concretely as the set of all formal expressions

$$a_2^{n_2} a_4^{n_4} a_1^{n_1} a_3^{n_3}$$

where $0 < n_1 < 3^{2m+j-i+2}$, $0 \leq n_2 < 3^{m+1}$, $0 \leq n_3 < 3^{2+i}$, $0 \leq n_4 < 3^2$. We define a product on these formal expressions (a formula derived from the relations of $C(j, i, m)$) by setting

$$a_2^{n_2} a_4^{n_4} a_1^{n_1} a_3^{n_3} * a_2^{m_2} a_4^{m_4} a_1^{m_1} a_3^{m_3} \quad = \quad a_2^{u_2} a_4^{u_4} a_1^{u_1} u_3^{u_3}$$

where

$$\begin{aligned} u_2 &= n_2 + m_2 - 3^m n_3 m_2 \\ u_4 &= n_4 + m_4 - 3^{m-i} n_3 m_2 + 3 n_1 m_2 \\ u_1 &= n_1 + m_1 - 3^{m+j-i+1} n_1 m_2 + 3^{2m+j-2i} n_3 m_2 - 3^{2m+j-i} n_3 m_4 \\ &\quad + 3^{2m+j-i+1} n_1 n_3 m_2 + 3^{3m+j-2i} n_3 m_2 \\ u_3 &= n_3 + m_3 + 3^{m+j-i} n_3 m_2 + 3^{i+1} n_3 m_1 - 3^{j+2} n_1 m_2 \end{aligned}$$

and where the exponents of a_1, a_2, a_3, a_4 are calculated modulo $3^{2m+j-i+2}$, $3^{2+i}, 3^2$. Tedious but straightforward calculations show that we get a group which satisfies all the relations of $C(j,i,m)$. So $C(j,i,m)$ has exactly $n(G) = 3^{3m+j+7}$ elements and also x_1 has order $3^{2m+j-i+2}$. As $G^{3^{2m+j-i+1}} = \langle x_1^{3^{2m+j-i+1}} \rangle$ and $[x_1, x_2, x_3] = [x_4^3, x_3] = x_1^{3^{2m+j-i+1}}$, it follows that the class is 3 and the exponent is $e(G) = 3^{2m+j-i+2}$. Also the exponent of $G/Z(G)$ is $a(G) = 3^{m+1}$. Since m, j and i are determined by $a(G)$, $n(G)$ and $e(G)$, the map $(j,i,m) \mapsto (e(G), n(G), a(G))$ is injective and the groups in the list are thus pairwise non-isomorphic.

It remains to establish the minimality of $C(j,i,m)$. Let H/K be a section of $G = C(j,i,m)$ that is powerful and of class 3. As $\gamma_3(H) \neq \{1\}$ and G is powerful, we must have elements $g_1, g_2, g_3 \in H$ of the following form

$$\begin{aligned} g_1 &= x_1 x_4^{r_1} \\ g_2 &= x_2 x_4^{r_2} \\ g_3 &= x_3 x_4^{r_3}. \end{aligned}$$

Now for H/K to remain of class 3, we can't have $x_1^{3^{2m+j-i+1}} \in K$. Let us see what restrictions this makes on K. Suppose

$$g = x_1^{n_1} x_2^{n_2} x_3^{n_3} x_4^{n_4}$$

is in K. Firstly as $[g, g_1, g_2] = x_1^{-3^{2m+j-i+1} n_3}$ is in K, we must have $3|n_3$. As $[g, g_1]^3$, $[g, g_2]^3 \in K$, we get that $[x_2, x_1]^{3n_1}, [x_2, x_1]^{3n_2} \in K$. It follows that 3^m divides n_1, n_2. Then considering $[g, g_3] \in K$ we see that 9 divides n_4 and g is of the form

$$g = x_1^{3^m m_1} x_2^{3^m m_2} x_3^{3m_3}.$$

Raising g to the power 3^{m+j+1} we see that $x_1^{3^{2m+j-i+1}m_3} \in K$ and 3 must divide m_3. Next considering $[g, g_1], [g, g_2] \in K$, we see that 3 divides m_1, m_2 and so g must be a power of x_3. This means that if g is non-trivial we must have $x_1^{3^{2m+j-i+1}}$ in K. We have thus shown that $K = \{1\}$. Now as $[g_1, g_2] = x_4^3 x_3^{-3^{2+j}} \notin \langle g_1^3, g_2^3, g_3^3, x_4^9\rangle$ and as H is powerful we must have that $x_4 \in H$. It follows that $H = G$. This finishes the proof. $\square$

References

[1] A. Abdollahi and G. Traustason, On locally finite p-groups satisfying an Engel condition, *Proc. Amer. Math. Soc.* **130** (2002), 2827-2836.

[2] W. Burnside, On an unsettled question in the theory of discontinuous groups, *Quart. J. Pure Appl. Math.* **37** (1901), 230-238.

[3] W. Burnside, On groups in which every two conjugate operators are permutable, *Proc. Lond. Math. Soc.* **35** (1902), 28-37.

[4] C. Hopkins, Finite groups in which conjugate operations are commutative, *Am. J. Math.* **51**, (1929), 35-41.

[5] F. W. Levi, Groups in which the commutator operation satisfies certain algebraic conditions, *J. Indian Math. Soc.* **6** (1942), 87-97.

[6] A. Lubotzki and A. Mann, Powerful p-groups. I. Finite groups, *J. Algebra* **105** (1987), 484-505.

[7] M. Zorn, Nilpotency of finite groups, *Bull. Amer. Math. Soc.* **42** (1936), 485-486.